Generalized metamaterials: Definitions and taxonomy

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This article reviews the development of metamaterials (MM), starting from Newton's discovery of the wave equation, and ends with a discussion of the need for a technical taxonomy (classification) of these materials, along with a better defined definition of metamaterials. It is intended to be a technical definition of metamaterials, based on a historical perspective. The evolution of MMs began with the discovery of the wave equation, traceable back to Newton's calculation of the speed of sound. The theory of sound evolved to include quasi-statics (Helmholtz) and the circuit equations of Kirchhoff's circuit laws, leading to the ultimate development of Maxwell's equations and the equation for the speed of light. Be it light, or sound, the speed of the wave-front travel defines the wavelength, and thus the quasi-static (QS) approximation. But there is much more at stake than QSs. Taxonomy requires a proper statement of the laws of physics, which includes at least the six basic network postulates: (P1) causality (non-causal/acausal), (P2) linearity (non-linear), (P3) real (complex) time response, (P4) passive (active), (P5) time-invariant (time varying), and (P6) reciprocal (non-reciprocal). These six postulates are extended to include MMs.
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I. INTRODUCTION

Acoustic metamaterials (AMMs) are composite materials whose properties are strongly different from their component characteristics. Usually the engineered AMMs possess unusual bulk properties, which require characterization. The purpose of this report is to provide a precise definition of AMMs, to improve on the vernacular definition "materials that do not appear in nature." What is needed is a technical definition, based on the mathematical properties of metamaterials. Toward this goal we draw upon, and then extend, the six postulates of circuit theory as defined by Carlin and Giordano (1964). These extended nine postulates include the long-wave approximation (i.e., quasi-statics), guided waves (transmission line horn theory), Rayleigh reciprocity, the active vs passive (positive-real) impedance property, and a generalization of causality. Based on these definitions, a technical taxonomy of MM is proposed.

A. Some history

Sergei Alexander Schelkunoff, a Bell Labs mathematical physicist who wrote the first textbook on electromagnetic waves in 1943, seems to have been the first to develop the concept of a metamaterial (MM). The first practical acoustic metamaterial (AMM) appears to have been realized by Winston Kock at Bell Labs in the 1940s in his work in *artificial acoustic dielectrics*. Schelkunoff's EM approach was significantly more detailed than Kock's, because acoustic wave propagation is described by a scalar equation, while EM wave propagation is described by Maxwell's vector equations. However, it is possible to mix acoustic and electrical modes, creating a vector-based AMM medium (i.e., electro-magnetically AMMs), which is even more structurally complex (Malinovsky and Donskoy, 2012). A brief review of the many historical breakthroughs related to AMMs should help guide the reader toward a deeper understanding of this subject.

Sir Issac Newton in 1686 set the first stage for acoustics by calculating the speed of sound. His formula was in error by $\sqrt{1.4}$, due to his ignorance of adiabatic compression of air, but his principles were otherwise correct. Following Newton's development of the wave equation, his contemporary Jean-Baptiste le Rond d'Alembert (1717–1783, see Fig. 1) first described the general solution for 1-dimensional waves in terms of the sum of any two arbitrary fixed functions traveling forward and backward in space, at the speed of sound. This formulation demonstrated the wave equation's linearity property, and the causality of acoustic systems. The key was the finite wave speed, which defined a wavelength for a pure tone as $\lambda f = c$. The relationship between wavelength and the speed of sound had been elucidated previously by Galileo Galilei and Marin Mersenne (1630), and even earlier by the ancient Greeks.

In Newton's day it was common to use a Taylor series to represent functional solutions to a differential equation. The exponential function e^{-t} had yet to be defined. Complex numbers had been formalized by Rafel Bombelli, but were not in common usage, even by Newton, who called them "impossible numbers" (Stillwell, 2010, p. 117). The Bernoulli brothers, Jacob and Johann, along with Johann's son Daniel, dramatically transformed mathematics by studying equations

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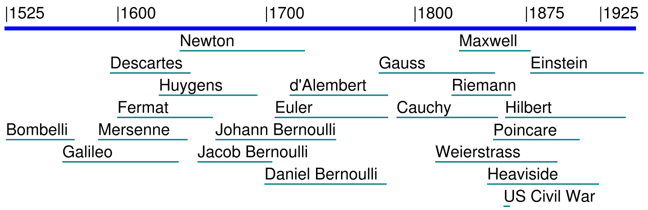

FIG. 1. (Color online) Time line showing some important mathematicians who lived between the 16th and 20th century.

defined by the Taylor series, such as the logarithmic function and its inverse, the exponential function. However it was Johann's student Leonhard Euler (1707–1783) who first took these studies to the next level, with results that went well beyond the Bernoullis' (Stillwell, 2010, p. 282). Carl Friedrich Gauss and Augustin-Louis Cauchy, but most importantly Georg Friedrich Bernhard Riemann (1851), next developed "complex" calculus, following in the footsteps of Newton and Euler's "real" calculus. With the introduction of the complex exponential function e^{-st}, where $s = \sigma + j\omega$ is the *complex radian frequency*, it was possible to deal with linear differential equations in the frequency domain. These concepts defined the frequency domain, allowing for the full development of the quasi-static (QS) approximation of impedance. In the words of Stillwell (p. 276),

> This resolution of the paradox of $\sqrt{-1}$ was so powerful, unexpected, and beautiful that only the word "miracle" seems adequate to describe it.

This quote is a testament to the power of the Laplace transform and the utility of complex-analytic functions of complex frequency.

The QS approximation was cleverly utilized by Hermann von Helmholtz (1821–1894), with his introduction of the *Helmholtz resonator*, based on first degree polynomial approximations of a Taylor series approximation of a transmission line, following naturally from d'Alembert's solution to the wave equation and Euler's complex exponential function. Soon after Helmholtz's discoveries, Gustav Kirchhoff's (1824–1887) circuit laws (1845) were introduced, 17 years before James Clerk Maxwell's electromagnetism (EM) equations were first written down. In 1863, at the same time the United States' Civil war was being fought, Maxwell first explained the speed of light with his demonstration of the EM vector wave equation, following in the footsteps of Newton's acoustic wave equation.

It was the invention of the telephone by Bell in 1876 that lead to the next major developments. This success began a technical revolution, driving massive innovation, much of it in electronics, including transmission line theory and the transistor. The first wave filter, used for telephonic transmission, was invented by George Ashley Campbell (1903) at the AT&T Development and Research (DR) department (Fagen, 1975; Millman, 1984). The concept of the *positive real* impedance function was defined by Otto Brune (1931), which identified the mathematical principle behind conservation of energy (Brune, 1931; Hunt, 1954; Van Valkenburg, 1960, 1964).

The development of modern radar began in the 1930s, and accelerated during World War II (1939–1945). Transmitting EM waves using metal wave-guides by Rayleigh (1897) gave the first hint of the expanded use of the QS approximation (Ramo *et al.*, 1965; Schelkunoff, 1943). Quarter-wave "stubs" on wave-guides acted as filters, making it possible to manipulate the frequency content of signals within the wave-guide. This development lead to electronics, at first in the form of traveling wave amplifiers, wave circulators, and directional couplers (Montgomery *et al.*, 1948), soon followed by the development of the transistor.

One important simplifying assumption in both acoustics and EM theory, is that of *guided waves,* where waves propagate mainly in the "longitudinal" direction, and are in *cutoff* in the transverse direction (i.e., perpendicular to the direction of propagation). In this situation the QS model applies in the transverse direction, but not in the longitudinal propagating direction. This leads to transmission line theory, critically important for the development of MMs. When the QS approximation is also imposed in the longitudinal direction, one has the case of the lumped element transmission line approximation (Campbell, 1903), widely used in engineering (Ramo *et al.*, 1965, p. 44, Appendix IV).

When the vector wave equation can be reduced to a scalar transmission line (TL) equation, the complexity of a problem may be greatly reduced. The most general form of the TL equation is the *Webster horn equation* (Gupta *et al.*, 2012; Pierce, 1981; Webster, 1919), which is the common assumption in guided acoustic transmission lines (horn theory), the ideal tool for MMs. In EM theory the horn equation is equivalent to the *telegrapher's equations* having a spatially varying characteristic impedance and wave speed. Given the driving point impedance looking into a horn, one may reconstruct the area function by solving a certain integral equation (Sondhi and Gopinath, 1971; Youla, 1964). These topics, while relevant to MMs, are well beyond the scope of the present discussion.

This study introduces a taxonomy to describe nonstandard effects that are observed in AMMs. An AMM can be redefined as an assembly of single-material transducers. To characterize the transducer "bundle" (a unit element of the AMM), we assume the wavelength is much larger than the component transducers. Thus we can use the quasi-static approximation to characterize the total AMM, using network and transmission line theory. The taxonomy proposed is based on well-known postulates. We believe this can lead to a major step forward when analyzing and characterizing

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TABLE I. There are several ways of indicating the quasi-static (QS) approximation. For network theory there is only one lattice constant a, which must be much less than the wavelength (wavelength constraint). These three constraints are not equivalent for MMs, when a is a three-vector of lattice constants, where the object may be a larger structured medium, spanning many wavelengths, but with a cell structure size much less than the wavelength. For example, each cell could be a Helmholtz resonator, or an EM transducer. Thus MMs require a generalized QS definition.

Measure	Domain
$ka < 1$	Wavenumber constraint
$\lambda > 2\pi a$	Wavelength constraint
$f_c < c/2\pi a$	Bandwidth constraint

Formally, QS is defined as $ka < 1$, where $k = 2\pi/\lambda = \omega/c$ and a is the cellular dimension or the size of the object (k and a can be vectors). Schelkunoff (1943) may have been the first to formalize this concept (but not the first to use it, as exemplified by the Helmholtz resonator). George Ashley Campbell was the first to use the concept in the important application of a wave-filter, 40 years before Schelkunoff (Campbell, 1903). These two men both worked for the telephone company (after 1929, called AT&T Bell Labs) (Fagen, 1975).

There are alternative definitions of the QS approximation, and for the case of MMs these may not be equivalent, depending on the cell structure. The alternatives are outlined in Table I.

III. CONCLUSIONS

We propose the general definition of a MM as a lattice of transducers, each of which follow postulates P1–P9, that include all transduction modalities of interest. A transducer converts between modalities. There are at least three different coupled modalities when dealing with MMs, including electrical, mechanical, and acoustical (there may be others). Furthermore it is necessary to generalize the concept of the QS approximation (P9) to allow for guided waves.

Given the combination of the important QS approximation, along with these space–time, linearity, and reciprocity properties, a rigorous definition and characterization of MMs can thus be established (or at least outlined). It is based on a taxonomy of such materials, formulated in terms of material and physical properties and in terms of extended network postulates.

Carlin's six network postulates have been extended to guide a better understanding of MMs by providing a taxonomy. One can classify MMs in terms of these nine postulates. A single transducer is not a MM; however, a lattice of transducers forms a MM; which has, for example, nonlinear, non-reciprocal, and active properties under the QS (or band-limited) approximation. For example Popa and Cummer designed a non-reciprocal active AMM cell unit, where they combined many non-MM units, such as a piezo-electric transducer, electrical components, and acoustic Helmholtz cavities.

The most important aspect of this current work is that it provides a foundation for characterizing MMs by using wave-based network postulates. This study refines the taxonomy of MMs based on a theoretical foundation. To our knowledge, this is the first attempt to rigorously classify MMs.

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AMMs, by providing deeper insights into their transmission properties.

II. THE DEFINITION AND TAXONOMY OF MMS

In either case, acoustic or EM, the definition of MMs seems fuzzy. MMs are frequently defined as those not found in nature:

> Metamaterials are materials engineered to have properties that have not yet been found in nature. As defined by Wikipedia.

This definition simply will not do, since "what may be found in nature" depends on what we know today. For example, are semiconductors MMs? I suspect many would say, "no." Yet nature does not provide us with silicon crystal wafers, from 6 (cm) ingots. More importantly this definition is starkly void of technical utility. Thus we will examine examples of MMs to develop a definition and eventual taxonomy.

During World War II, submarines would sit at a depth that allowed them to be invisible to sonar waves caused by a "refraction" of the acoustic waves, producing a region of total isolation from the surface sonar. A whale uses the same effect (denoted as a SOFAR channel) to communicate over many thousands of miles. Again, it looks like an apparent miracle; yet it is provided by nature. Both of these are great examples of an important (and also practical) cloaking effect that is simply due to a temperature gradient that then changes the speed of sound, thus creating the SOFAR channel. Norris (2008) explains the physics behind acoustic cloaking by deriving the acoustic cloaking theory using the transformation between the physical space to the virtual space. Efforts realizing and optimizing cloaks on AMMs were recently published (Cai, 2012; Gokhale *et al.*, 2012).

In the EM case, materials have been synthesized that have a negative μ (permeability) and ε (permittivity) (Orfanidis, 2008, Sec. 8.6). In the acoustics case this would imply a negative mass (Huang and Sun, 2012) and bulk modulus (stiffness). Such properties can only exist over a narrow frequency range, due to causality. Every causal complex analytic function, such as an impedance and the propagation function [i.e., complex wave number $\kappa(s) = \beta(s) + jk(s)$] must obey the Cauchy–Riemann conditions, which couple the real and imaginary parts of every analytic function. This is an important extension of the Kramers–Kronig (i.e., Hilbert Transform) relations used in the Fourier domain (Brillouin, 1960, Chap. III, p. 43–83).

As with the first definition, the attempt to categorize MMs seems to be neither technical nor obvious. Therefore the objective of this study is a mathematical statement of MM properties, discussed in Sec. II B.

A. Carlin's network postulates

Since the QS approximation is a key principle in the design of MMs, QS network theory is well-suited to characterize MMs. Thus, as a starting point, MMs are conveniently cataloged in terms of the six postulates of QS network theory (Carlin and Giordano, 1964): a network may be characterized as (P1) *causal* (or non-causal/acausal), (P2) *linear* (or non-linear), (P3) *having a real* (or complex) time response, (P4) *passive* (or active), (P5) *time-invariant* (or time varying) and (P6) *reciprocal* (or non-reciprocal). These postulates are widely accepted in the network theory literature (Ramo *et al.*, 1965; Van Valkenburg, 1960, 1964).

These six postulates describe the properties of a system having an input and an output. For the case of an electromagnetic transducer (loudspeaker) the system is described by a two-port network as shown in Fig. 2. P6 is inherently a two-port network property, while P1–P5 apply to one-port networks as well (e.g., a driving point impedance is a one-port). For example the electrical impedance of a loudspeaker is defined by

$$Z_e(s) = \frac{\Phi(\omega)}{I(\omega)}\bigg|_{U=0}.$$

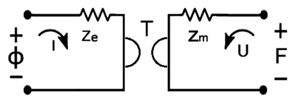
Note that the driving point impedance is a causal function, thus it has a Laplace transform and is a function of the complex frequency $s = \sigma + j\omega$, whereas the Fourier transforms of the voltage $\Phi(\omega)$ and current $I(\omega)$ are functions of the real radian frequency ω.

1. Transmission matrix

The two-port *transmission matrix* for an acoustic transducer (loudspeaker) is defined as

$$\begin{bmatrix}\Phi_i\\ I_i\end{bmatrix} = \begin{bmatrix}A(s) & B(s)\\ C(s) & D(s)\end{bmatrix}\begin{bmatrix}F_l\\ -U_l\end{bmatrix} = \frac{1}{T}\begin{bmatrix}Z_e(s) & Z_e(s)z_m(s)+T^2\\ 1 & z_m(s)\end{bmatrix}\begin{bmatrix}F_l\\ -U_l\end{bmatrix}. \quad (1)$$

In the above example the input is electrical (voltage and current) $[\Phi_i, I_i]$ and the output (load) are the mechanical (force and velocity) $[F_l, U_l]$. The first matrix is the general case, expressed in terms of four unspecified functions $A(s), B(s), C(s), D(s)$, while the second matrix is for the specific example



$$\begin{bmatrix}\Phi\\ I\end{bmatrix} = \begin{bmatrix}1 & Z_e\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & T\\ T^{-1} & 0\end{bmatrix}\begin{bmatrix}1 & z_m\\ 0 & 1\end{bmatrix}\begin{bmatrix}F\\ -U\end{bmatrix}$$

FIG. 2. A schematic representation of a two-port ABCD electro-mechanic system using Hunt parameters $Z_e(s)$, $z_m(s)$, and $T(s)$: electrical impedance, mechanical impedances, and transduction coefficient (Hunt, 1954; Kim and Allen, 2013). Also $\Phi(f)$, $I(f)$, $F(f)$, and $U(f)$ are the frequency domain voltage, current, force, and volume velocity, respectively. Notice how the ABCD matrix method "factors" the two-port model into 2×2 matrices. This allows one to separate the modeling from the algebra. The flow is always defined into the port. Thus it is necessary to force a negative sign on the volume velocity.

of Fig. 2. The four entries are the electrical driving point impedance $Z_e(s)$, the mechanical impedance $z_m(s)$, and the transduction $T = B_0 l$, where B_0 is the zero frequency (aka DC) magnetic flux strength and l is the length of the wire crossing the flux. Since the transmission matrix is anti-reciprocal, its determinate $\Delta_T = -1$, as is easily verified.

Other common transduction examples of cross-modality transduction include current–thermal (thermoelectric effect) and force–voltage (piezoelectric effect). These systems are all reciprocal, thus the transduction has the same sign.

2. Impedance matrix

The corresponding two-port *impedance matrix* is

$$\begin{bmatrix} \Phi_i \\ F_l \end{bmatrix} = \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_i \\ U_l \end{bmatrix} = \begin{bmatrix} Z_e(s) & -T(s) \\ T(s) & z_m(s) \end{bmatrix} \begin{bmatrix} I_i \\ U_l \end{bmatrix}. \quad (2)$$

The impedance matrix is an alternative description of the system but with generalized forces $[\Phi_i, F_l]$ on the left and generalized flows $[I_i, U_l]$ on the right. A rearrangement of the equations allows one to go from one set of parameters to the other (Van Valkenburg, 1960). Since the electromagnetic transducer is anti-reciprocal, $z_{21} = -z_{12} = T = B_0 l$.

B. Modified, expanded, and additional postulates

Our definition of MMs must go beyond postulates P1–P6, since MMs result from the interaction of waves in a structured medium, along with other properties not covered by classic network theory (e.g., the quantum Hall effect). Assuming QS, the wavelength must be large relative to the medium's lattice constants. Thus the QS property must be extended for MM to three dimensions, and possibly to the cases of an-isotropic and random media.

1. Causality (P1)

As stated above, due to causality the negative properties (e.g., negative refractive index) of AMMs must be limited in bandwidth, as a result of the Cauchy–Riemann conditions. However even causality needs to be extended to include the delay, as quantified by the d'Alembert solution to the wave equation, which means that the delay is proportional to the distance. Thus we generalize P1 to include the space dependent delay. When we wish to discuss this property we denote it *Einstein causality*, which says that the delay must be proportional to the distance x, with impulse response $\delta(t - x/c)$.

2. Linearity (P2)

The wave properties of MMs may be non-linear. This is not restrictive, as most physical systems are naturally non-linear. For example, a capacitor is inherently non-linear: as the charge builds up on the plates of the capacitor, a stress is applied to the intermediate dielectric due to the electrostatic force $F = qE$. In a similar manner, an inductor is non-linear. Two wires carrying a current are attracted or repelled, due to the force created by the flux. The net force is the product of the two fluxes due to each current.

Most physical systems are naturally non-linear, it is simply a matter of degree. An important example is an amplifier with negative feedback, with very large open-loop gain. There are many types of non-linearity such as instantaneous distortion, or time-based "memory" dependence (e.g., hysteresis). The linear property is so critical for analysis (Quan *et al.*, 2012), that linear approximations are often used whenever possible.

3. Positive-realness and conservation of energy (P4)

We greatly extend P4 (passive/active) by building in the physics behind conservation of energy: Otto Brune's *positive real* (PR) or "physically realizable" condition for a passive system. Following up on the earlier work of his primary Ph.D. thesis advisor Wilhelm Cauer (1900–1945), and working with Norbert Weiner and Vannevar Bush at MIT, Otto Brune mathematically characterized the properties of every PR one-port driving point impedance.

Given any PR impedance $Z(s) = R(\sigma,\omega) + jX(\sigma,\omega)$, having real part $R(\sigma,\omega)$ and imaginary part $X(\sigma,\omega)$, the impedance is defined as being PR (Brune, 1931) if and only if

$$R(\sigma \geq 0, \omega) \geq 0. \quad (3)$$

That is, the real part of any PR impedance is non-negative everywhere in the right half s plane ($\sigma \geq 0$). This is a very strong condition on the complex analytic function $Z(s)$ of a complex variable s. This condition is equivalent to any of the following statements: (1) there are no poles or zeros in the right half plane [$Z(s)$ may have poles and zeros on the $\sigma = 0$ axis], (2) if $Z(s)$ is PR then its reciprocal $Y(s) = 1/Z(s)$ is PR (the poles and zeros swap), (3) if the impedance may be written as the ratio of two polynomials (a limited case) having degrees N and L, then $|N - L| \leq 1$, (4) the angle of the impedance $\theta \equiv \angle Z$ lies between $[-\pi \leq \theta \leq \pi]$, and (5) the impedance and its reciprocal are *complex analytic* in the right half plane, thus they each obey the Cauchy–Riemann conditions.

The PR condition assures that every impedance is *positive-definite* (PD), thus guaranteeing conservation of energy is obeyed (Schwinger and Saxon, 1968, p.17). This means that the total energy absorbed by any PR impedance must remain positive for all time, namely,

$$\mathcal{E}(t) = \int_{-\infty}^{t} v(t) i(t)\, dt = \int_{-\infty}^{t} i(t) \star z(t)\, i(t)\, dt > 0,$$

where $i(t)$ is any current, $v(t) = z(t) \star i(t)$ is the corresponding voltage and $z(t)$ is the real causal impulse response of the impedance [related by the Laplace transform $z(t) \leftrightarrow Z(s)$]. In summary, if $Z(s)$ is PR, $\mathcal{E}(t)$ is PD.

As discussed in detail by Van Valkenburg, any rational PR impedance can be represented as a *rational polynomial fraction expansion* (residue expansion), which can be expanded into first-order poles as

$$Z(s) = K \frac{\prod_{i=1}^{L}(s - n_i)}{\prod_{k=1}^{N}(s - d_k)} = \sum_n \frac{\rho_n}{s - s_n} e^{j(\theta_n - \theta_d)}, \quad (4)$$

where ρ_n is a complex scale factor (residue). Every pole in a PR function has only simple poles *and* zeros, requiring that $|L - N| \leq 1$ (Van Valkenburg, 1964).

Whereas the PD property clearly follows P3 (conservation of energy), the physics is not so clear. Specifically what is the physical meaning of the specific constraints on $Z(s)$? In many ways, the impedance concept is highly artificial.

When the impedance is not rational, special care must be taken. An example of this is the semi-inductor $M\sqrt{s}$ and semi-capacitor $K/\sqrt{s}$ due, for example, to the *skin effect* in EM theory and viscous and thermal losses in acoustics, both of which are frequency dependent boundary-layer diffusion losses. They remain positive-real but have a branch cut, thus are double valued in frequency.

4. Rayleigh Reciprocity (P6)

Reciprocity is defined in terms of the unloaded output voltage that results from an input current. Specifically

$$\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} = \frac{1}{C(s)} \begin{bmatrix} A(s) & \Delta_T \\ 1 & D(s) \end{bmatrix}, \quad (5)$$

where $\Delta_T = A(s)D(s) - B(s)C(s) = \pm 1$ for the reciprocal and anti-reciprocal systems, respectively. This is best understood in terms of Eq. (2). The off-diagonal coefficients $z_{12}(s)$ and $z_{21}(s)$ are defined as

$$z_{12}(s) = \left.\frac{\Phi_i}{U_l}\right|_{I_i=0} \quad z_{21}(s) = \left.\frac{F_l}{I_i}\right|_{U_l=0}.$$

If these off-diagonal elements are equal [$z_{12}(s) = z_{21}(s)$] the system is said to obey *Rayleigh reciprocity*. If they are opposite in sign [$z_{12}(s) = -z_{21}(s)$], the system is said to be *anti-reciprocal*. If a network has neither reciprocal nor anti-reciprocal characteristics, then we denote it as *non-reciprocal* (McMillan, 1946). The most comprehensive discussion of reciprocity, even to this day, is that of (Rayleigh, 1896, Vol. I, pp. 150–160). The reciprocal case may be modeled as an ideal transformer (Van Valkenburg, 1964) while for the anti-reciprocal case the generalized force and flow are swapped across the two-port. An electromagnetic transducer (e.g., a moving coil loudspeaker or electrical motor) is anti-reciprocal (Beranek and Mellow, 2014; Kim and Allen, 2013), requiring a gyrator rather than a transformer, as shown in Fig. 2.

Non-reciprocity (or a lack of reciprocity) is another major characteristic of AMMs (Maznev *et al.*, 2013; Popa and Cummer, 2014; Sounas *et al.*, 2013; Fleury, 2015). As discussed by Popa and Cummer (2014), the on-going debate on naming a unidirectional acoustic device indicates the need for better taxonomy in the field of AMMs. Unidirectional acoustic devices have many names, such as acoustic diodes, rectifiers, isolators, and non-reciprocal media. All these terms describe an AMM that breaks the directional symmetry property of transmission, like an operational amplifier does.

5. Reversibility (new P7)

A reversible system is invariant to the input and output impedances being swapped. This property is defined by the input and output impedances being equal.

Referring to Eq. (5), when the system is *reversible* $z_{11}(s) = z_{22}(s)$ and, in terms of the transmission matrix variables, $[A(s)/C(s)] = [D(s)/C(s)]$ or simply $A(s) = D(s)$ if $C(s)$ is not zero.

An example of a non-reversible system is a transformer where the turns ratio is not one. An ideal operational amplifier (when the power is turned on) is also non-reversible due to the large impedance difference between the input and output. Furthermore it is *active* (it has a power gain, due to the current gain at constant voltage) (Van Valkenburg, 1960).

Generalizations of this lead to group theory, and *Noether's theorem*. These generalizations apply to systems with many modes whereas MMs operate below a cutoff frequency (Chesnais *et al.*, 2012), meaning that like the case of the TL, there are no propagating transverse modes. While this assumption is never exact, it leads to highly accurate results because the non-propagating evanescent transverse modes are attenuated over a short distance, and thus, in practice, may be ignored (Montgomery *et al.*, 1948; Orfanidis, 2008; Schwinger and Saxon, 1968, Chaps. 9–11).

We extend the Carlin and Giordano postulates to include reversibility (P7), which was refined by Van Valkenburg (1964). To satisfy the reversibility condition, the diagonal components in a system's impedance matrix must be equal. In other words, the input force and the flow are proportional to the output force and flow, respectively (i.e., $Z_e = z_m$).

6. Spatial dependencies of MM (new P8)

The characteristic impedance and wave speed of MMs may be strongly frequency and/or spatially dependent, or even be negative over some limited frequency ranges. Due to *causality*, the concept of a negative group velocity must be restricted to a limited bandwidth (Brillouin, 1960, Chap. II, p. 19–23). As is made clear by Einstein's theory of relativity, all materials must be strictly causal (P1), a view that must therefore apply to acoustics, but at a very different time scale.

7. The QS constraint (new P9)

An important property of MM is the use of the QS approximation, especially when the waves are guided or band limited. This property is not mentioned in the six postulates of Carlin and Giordano (1964). Only when the dimensions of a cellular structure in the material are much less than the wavelength, can the QS approximation be valid. This effect can be viewed as a *mode filter* that suppresses unwanted (or conversely enhances the desired) modes. But a single number used to quantify the structure is not adequate for MMs, as it can be a three dimensional specification, as in a semiconductor lattice, or even a random variable matrix.